Global positioning of spin GPS scheme for half-spin massive spinors[⋆]

S. Jadach[1,2], B.F.L. Ward[1,3,4], Z. Wąs[1,2]

[1] CERN, Theory Division, 1211 Geneva 23, Switzerland
[2] Institute of Nuclear Physics, ul. Kawiory 26a, Kraków, Poland
[3] Department of Physics and Astronomy, The University of Tennessee, Knoxville, TN 37996-1200, USA
[4] SLAC, Stanford University, Stanford, CA 94309, USA

Abstract. We present a simple and flexible method of keeping track of the complex phases and spin quantisation axes for half-spin initial- and final-state Weyl spinors in scattering amplitudes of Standard Model high energy physics processes. Both cases of massless and massive spinors are discussed. The method is demonstrated and checked numerically for spin correlations in $\tau\bar{\tau}$ production and decay. Its application is in our work of combining effects due to multiple photon emission (exponentiation) and spin, embodied in the Monte Carlo event generators for production and decay of unstable fermions such as the τ lepton, t-quark and hypothetical new heavy particles. In particular, the recurrent problem of combining, for such unstable fermions, one author's calculation of production and another author's calculation of decay, in the presence or absence of multiple photon effects, is there given a practical solution, both for Weyl spinor methods and for the traditional Jacob-Wick helicity methods. Moreover, for massive fermions we give a simple representation of the amplitude for $n(\gamma)$ emission ideally suited for numerical evaluation. No other method is known to us which for arbitrary n has been realized numerically for unstable, massive fermions. Our paper can contribute also, to the discussion on design principles of the phenomenology work for the future accelerators such as LHC or NLC.

1 Introduction: What is the problem?

Weyl spinors (Ws) are used in very powerful algorithms to calculate multiple-particle and in particular multiple-photon spin amplitudes [1,2]. In Ws calculations the complete control of the spin-dependent complex phase, i.e. of spin quantization frames, is usually neglected. This is not a problem, when one is interested in cross sections for stable particles, but it becomes a crucial limitation if one needs to combine them with the spin amplitudes for the decay of an almost stable (narrow resonance) fermion.

The complete spin amplitude for production and decay is given by the formula:

$$\mathcal{M} = \sum_{\lambda=\pm 1/2} \mathcal{M}_\lambda^{\text{prod.}} \mathcal{M}_\lambda^{\text{decay}}, \qquad (1)$$

where λ is the projection of the *intrinsic* spin on the z-axis in a certain fermion rest frame (see discussion in the

appendix of [3], and also [4]). Let us call this particular fermion rest frame the spin quantisation reference frame, SQRF. When, in the above formula, both production and decay spin amplitudes are obtained from Feynman rules in *the same* calculation, with the same intermediate fermion states $u(p, \lambda)$, the situation is simple – we even do not need to know very precisely the SQRF's. The above is not true in the many important practical cases such as the production of the τ lepton, t-quark and hypothetical supersymmetric particles. The reason is the following: the fermion may have many decay modes, with complicated multiparticle final states, and complicated half-phenomenological decay matrix elements, calculated and implemented separately. For instance in the case of τ decay, the TAUOLA Monte Carlo [5] represents by itself an independent computational project. In fact every one out of more than twenty decay modes of the τ represents an independent project both from the computational and from the theoretical perspective. In such a case, it has been a recurrent problem in the literature and in the use of the literature by a third party that one author's calculation of $\mathcal{M}_\lambda^{\text{prod.}}$ and another author's calculation of $\mathcal{M}_\lambda^{\text{decay}}$ do not really occur in the same SQRF, so that the blind substitution into (1)

gives wrong results. In our paper, we will present a possible solution to this problem: The third party can transcribe both $\mathcal{M}_\lambda^{\text{prod.}}$ and $\mathcal{M}_\lambda^{\text{decay}}$ according to our (GPS) rules given below[1].

For these practical reasons, combining production and decay at the *density matrix* level, even if it seems at first more complicated, is worth an effort, because formula (1) would lead to independent sets of spin amplitudes for distinct combinations of decay modes of unstable final-state fermions. The density matrix formalism allows for clean modularity in the calculations and in the architecture of the corresponding Monte Carlo event generators.

The price one has to pay (for spin effects not to be destroyed) is that the complete combined density matrix for all final-state unstable particles must be provided by the production program with the careful definition of the corresponding SQRFs. That requires, in particular, full control of the relative phases of all production spin amplitudes.

In our paper we shall rely on massless and massive Weyl spinors defined in two papers of Kleiss and Stirling (KS) [6,7]. An analogous set of massless spinors is also defined in [8]. By supplementing these techniques with rules for complete control of spin quantization axes (complex phases) we turn them into a powerful tool for calculations of unstable half-spin fermions, useful for a Monte Carlo simulation of production and decay. The aim of the present work is to adapt the spin amplitude techniques to the framework established in the works [5,9,10], see also [11,12]. We shall use it in our work on exclusive coherent exponentiation [13].

Let us start with the fundamental question of defining properly the spin quantization axes (and complex phases) for half-spin Weyl spinors. We present details of the lowest-order τ-pair production process

$$e^-(p_1,\lambda_1) + e^+(p_2,\lambda_2) \rightarrow \tau^-(q_1,\mu_1) + \tau^+(q_2,\mu_2) \quad (2)$$

with spin amplitudes $\mathcal{M}_{\lambda_1\lambda_2\mu_1\mu_2}^{\text{prod}}$ where $\mu_1,\mu_2 = \pm 1$ denote twice the spin z-projection of the corresponding τ lepton in its rest frame, and λ_i are analogous spin indices of $e^\pm$. To be clear, we will not necessarily always restrict ourselves to λ_i and μ_i being the conventional Jacob-Wick helicities [14].

The full differential cross section for the combined τ-pair production and decay reads as follows

$$d\sigma = d\sigma^{\text{prod}}(q_1,q_2)\, \frac{d\Gamma_1^{\text{decay}}(\tau^- \rightarrow X_1)}{\Gamma_1^{\text{decay}}}$$
$$\times \frac{d\Gamma_2^{\text{decay}}(\tau^+ \rightarrow X_2)}{\Gamma_2^{\text{decay}}}$$
$$\times \left(\sum_{a,b=0}^{3} \varepsilon_1^a \varepsilon_2^b R_{abcd}\, h_1^c h_2^d \right) \quad (3)$$

where $d\sigma^{\text{prod}}$ is the unpolarized differential cross section for the production process only, and $d\Gamma_i^{\text{decay}}$ are unpolarized differential partial widths for decays of the $\tau^\pm$. The *polarimeter* vectors h_i^a are related to the τ decay matrix element (see (2.18) of [15] for its precise definition)[2], and ε_i are conventional spin polarization vectors of $e^\pm$ parametrizing their spin density matrices $\rho_{\lambda,\bar\lambda} = (1 + \vec{\sigma}\cdot\vec{\varepsilon})_{\lambda,\bar\lambda}/2$.

All spin effects are embodied in the last term, which includes the full spin density matrix related to the production spin amplitudes as follows:

$$R_{abcd} = N^{-1} \sum_{\lambda_i,\mu_i,\bar\lambda_i,\bar\mu_i} \mathcal{M}_{\lambda_1\lambda_2\mu_1\mu_2}(\mathcal{M}_{\bar\lambda_1\bar\lambda_2\bar\mu_1\bar\mu_2})^\star$$
$$\times \sigma_{\lambda_1\bar\lambda_1}^a \sigma_{\lambda_2\bar\lambda_2}^b \sigma_{\bar\mu_1\mu_1}^c \sigma_{\bar\mu_2\mu_2}^d,$$
$$N = \sum_{\lambda_i,\mu_i} |\mathcal{M}_{\lambda_1\lambda_2\mu_1\mu_2}|^2, \quad a,b,c,d = 0,1,2,3, \quad (4)$$

where σ^k for $k=1,2,3$ are Pauli matrices and $\sigma_{\lambda,\mu}^0 = \delta_{\lambda,\mu}$ is just the unit matrix.

The translation in (4) from spin amplitudes to R_{abcd}, and the analogous translation from decay spin amplitudes to the vectors h_c and h_d, occurs in the SQRF of τ^- for indices $c, \mu_1, \bar\mu_1$ and in the SQRF of τ^+ for indices $d, \mu_2, \bar\mu_2$. Let us stress again that this translation occurs not in some arbitrary $\tau^\pm$ rest frame but in *precisely the same* SQRF of $\tau^\pm$, where μ_i was originally defined as a spin projection on the z-axis! Not only the z-axis has to be known, but also the x- and y-axes[3]. Being in the SQRF makes it also legitimate to use Pauli matrices for the translation of spin indices into vector indices in (4)[4].

In the practical calculations that implement the above methodology, see KORALB [10] and KORALZ [16], a special *subprogram* `TraLor` is available, which implements the Lorentz transformation from the SQRF[5] of each $\tau^\pm$ down to the laboratory system. It is used to transform the momenta of all $\tau^\pm$ decay products into the laboratory frame. This arrangement fully guarantees that all spin effects, including spin correlations, are properly reproduced in the Monte Carlo event generation.

Having convinced the reader that the knowledge and the proper use of the SQRF for each fermion is very important, let us look into how it is defined. The obvious

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[1] This point is of the special relevance for the design of the LHC software which is now taking place

[2] The vectors h_i^a depend, in general, on momenta of all τ decay products

[3] The change of SQRF by a rotation around the spin quantization z-axis by an angle ϕ introduces the factor $\exp((i\phi/2)\mu_1)$ (rotation in spinor representation) in the spin amplitudes and consequently an $R_3(\phi)$ rotation in normal space (i.e. adjoint to spinor representation) acting on the index c in R_{abcd}. N.B. a simplest example of Wick rotation is actually $R_3(\phi)$

[4] Pauli matrices play here the role of Clebsch-Gordan coefficients to combine two half-spin representations of the rotation group (two indices in the spin density matrix) into spin 0 and 1 angular momentum objects, see (A.10) in [15]

[5] The `TraLor` *subprogram* actually provides explicit definition of the SQRF frame

choice is to employ the standard definition of spin states by Jacob and Wick (JW) [14], the so-called helicity states. In [9] spinors of outgoing $\tau^{\pm}$ (and also of $e^{\pm}$ beams) were defined using a variant of the JW prescription[6]. In the JW method, states with definite spin projection on the z-axis (helicity states) are defined by construction in the SQRF; the full Lorentz transformation down to the laboratory frame is used as an inherent construction element in the definition of the spin state. In KORALB [9,10] the JW spin quantization principle was also successfully used to define spin states of $e^{\pm}$ and $\tau^{\pm}$ in the presence of one photon.

On the other hand, looking at the calculations of [9, 10] it is quite clear that the JW method is well suited to lowest order $e^-e^+ \to \tau^-\tau^+$, but that it is rather complicated to deal with already in the case $e^-e^+ \to \tau^-\tau^+\gamma$ since the orientation of SQRF systems depends on the photon direction even in the soft-photon limit. One definitely needs a better method to go beyond the single-photon case. The natural candidates are methods based on the Weyl spinors, which have proved to be very successful for the evaluation of multiple photon emission spin amplitudes for massless and massive spinors. Generally Weyl spinors (Ws) have their own internal definition of the SQRF, substantially different from that in the Jacob-Wick method. For example in the KS technique [6,7] the spin quantization axis in the rest frame of the massive fermion is related to some auxiliary light-like vector k_0 involved in the definition of the massive spinor. The typical definition of the spinor in the KS method involves a single "brutal" transformation from the reference fermion state into the state of definite mass, spin and momentum rather than the explicit "soft" rotation/boost method of the JW method. The problem is that in the transformation method of the Ws/KS techniques the phase, and therefore the position of the x- and y-axes, is lost or undefined. This is not a great problem for most of the practical applications of the Ws/KS techniques, such as the calculation of multiple photon emission spin amplitudes, which are usually done with the "modulo phase factor" approach anyway. This feature of Ws/KS methods inhibits, however, the use of this method for unstable fermions. The aim of this paper is to show how to restore the knowledge of the complex phase and the SQRF for the KS massive spinors of [6,7].

In the following we shall repeat many definitions of [6, 7] in detail. This is made necessary by the fact the central question is the control of the spinor phases, and even knowledge that this phase is zero (in certain cases) forms an important new result! Our final answer for the problem of control of Weyl spinor phases and all three axes of the SQRF is very simple – the reader should not be misled by its apparent simplicity, it is an important and new supplement of the existing Ws/KS methods.

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[6] The JW [14] recipe was modified in [9], and in fact this choice is closer to that in [3]. For final fermions the y-axis was placed in the reaction plane, while in the JW prescription it is related by direct rotation (around the axis perpendicular to the reaction plane) to the laboratory y-axis

2 Basics of the Weyl technique

For Dirac spinors, wherever an explicit representation is needed, we employ the Weyl representation of gamma matrices

$$\gamma^0 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad \gamma^k = \begin{bmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{bmatrix}, \quad \gamma^5 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}. \quad (5)$$

In the Weyl representation, spinors transform under rotations around the k-th axis and boosts along the j-th axis as follows [17]:

$$S(R_k(\phi)) = \exp\left(-\frac{i}{2}\phi \begin{bmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{bmatrix}\right),$$

$$S(B_j(\chi)) = \exp\left(\frac{1}{2}\chi \begin{bmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{bmatrix}\right). \quad (6)$$

We define eigenstates with definite *massless* four-momentum $p^2 = 0$ and chirality $\lambda = \pm 1$

$$\not{p}u_\lambda(p) = 0, \quad \omega_\lambda u_\lambda(p) = u_\lambda(p), \quad \omega_\lambda \equiv \frac{1}{2}(1 + \lambda\gamma_5), \quad (7)$$

with the normalization condition $u_\lambda(p)\bar{u}_\lambda(p) = \omega_\lambda \not{p}$. The four basic *massless* spinors are defined in a certain primary reference frame (PRF), as follows:

$$\mathfrak{u}_+(\zeta_\uparrow) = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathfrak{u}_+(\zeta_\downarrow) = \begin{bmatrix} 0 \\ \sqrt{2} \\ 0 \\ 0 \end{bmatrix},$$

$$\mathfrak{u}_-(\zeta_\uparrow) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\sqrt{2} \end{bmatrix}, \quad \mathfrak{u}_-(\zeta_\downarrow) = \begin{bmatrix} 0 \\ 0 \\ \sqrt{2} \\ 0 \end{bmatrix}, \quad (8)$$

and they correspond to particles flying along the z-axis: $\zeta_\uparrow = (1,0,0,1)$ and $\zeta_\downarrow = (1,0,0,-1)$. It is important to remember that the relative phases in the above set of basic vectors are constrained uniquely by the following relations

$$\mathfrak{u}_\pm(\zeta_\downarrow) = S(R_2(\pi))\mathfrak{u}_\pm(\zeta_\uparrow), \quad \mathfrak{u}_+(\zeta_\uparrow) = -\not{\eta}\mathfrak{u}_-(\zeta_\uparrow),$$

$$\mathfrak{u}_+(\zeta_\downarrow) = \not{\eta}\mathfrak{u}_-(\zeta_\downarrow), \quad (9)$$

where[7] $\eta = \hat{e}_x = (0,1,0,0)$, $\eta^2 = -1$, $\eta \cdot \zeta_\uparrow = 0$ and $\eta \cdot \zeta_\downarrow = 0$.

3 Massless spinors

The arbitrary massless spinor of momentum p and chirality λ is generated according to the KS method [6], out of

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[7] Our $\zeta_\downarrow$ corresponds to the vector called k_0 in [6] and η is denoted there by k_1

the two constant basic spinors of opposite chirality (two out of four in (8))

$$u_\lambda(p) = \frac{1}{\sqrt{2p \cdot \zeta}} \not{p} \mathfrak{u}_{-\lambda}(\zeta), \qquad \zeta \equiv \zeta_\downarrow. \qquad (10)$$

Written explicitly, see also [8], it looks as follows:

$$u_+(p) = \begin{bmatrix} \sqrt{p^+} \\ \sqrt{p^-}e^{i\phi} \\ 0 \\ 0 \end{bmatrix}, \quad u_-(p) = \begin{bmatrix} 0 \\ 0 \\ -\sqrt{p^-}e^{-i\phi} \\ \sqrt{p^+} \end{bmatrix},$$

$$p^\pm \equiv p^0 \pm p^3, \quad p^1 + ip^2 \equiv p_T e^{i\phi}. \qquad (11)$$

The above expressions lead directly to an explicit expression for the *inner product* of the two massless spinors

$$s_+(p_1, p_2) \equiv \bar{u}_+(p_1)u_-(p_2)$$
$$= \sqrt{p_1^- p_2^+}e^{-i\phi_1} - \sqrt{p_1^+ p_2^-}e^{-i\phi_2},$$
$$s_-(p_1, p_2) \equiv \bar{u}_-(p_1)u_+(p_2) = -(s_+(p_1, p_2))^*. \quad (12)$$

It is understood that the above formula is evaluated using four-momenta in the PRF, see (8). The same formula can be obtained using the original KS expression[8]:

$$s_+(p, q) = 2\frac{1}{\sqrt{2p\zeta}} \frac{1}{\sqrt{2q\zeta}} \qquad (13)$$
$$\times [(p\zeta)(q\eta) - (p\eta)(q\zeta) - i\epsilon_{\mu\nu\rho\sigma}\zeta^\mu\eta^\nu p^\rho q^\sigma],$$

which is applicable in any reference frame, not only in the PRF.

4 Massive spinors

Spinors for the massive particle with four momentum p (with $p^2 = m^2$) can also be defined using a construction similar to that in the massless case:

$$u(p, \lambda) = \frac{1}{\sqrt{2p \cdot \zeta}} (\not{p} + m)\, \mathfrak{u}_{-\lambda}(\zeta),$$

$$v(p, \lambda) = \frac{1}{\sqrt{2p \cdot \zeta}} (\not{p} - m)\, \mathfrak{u}_\lambda(\zeta). \qquad (14)$$

As we see, in order to obtain the v-spinor for the antiparticle we have to substitute $m \to -m$ and $\lambda \to -\lambda$, and this is a general rule in the following. Apart from the Dirac equation $(\not{p} - m)u(p, \lambda) = 0$ and $(\not{p} + m)v(p, \lambda) = 0$ the above spinors obey the following normalization and completeness relations:

$$u(p, \lambda)\, \bar{u}(p, \lambda) = \frac{1}{2}(1 + \lambda\gamma_5\not{s})\,(\not{p} + m),$$

$$v(p, \lambda)\, \bar{v}(p, \lambda) = \frac{1}{2}(1 + \lambda\gamma_5\not{s})\,(\not{p} - m), \qquad (15)$$

where

$$s = \frac{p}{m} - \zeta\frac{m}{p \cdot \zeta}, \quad s^2 \equiv -1, \qquad (16)$$

is the *spin quantization vector*; in the fermion rest frame, it points opposite to ζ, i.e. $\vec{s} \sim -\vec{\zeta}$, or in other words the spin quantization axis for every fermion is *guided* by the single and common massless vector ζ.

The definition of (14) can be rewritten explicitly in terms of massless spinors as follows

$$u(p, \lambda) = u_\lambda(p_\zeta) + \frac{m}{\sqrt{2p\zeta}}\mathfrak{u}_{-\lambda}(\zeta),$$

$$v(p, \lambda) = u_{-\lambda}(p_\zeta) - \frac{m}{\sqrt{2p\zeta}}\mathfrak{u}_\lambda(\zeta), \qquad (17)$$

where

$$p_\zeta = p - \zeta\frac{m^2}{2p \cdot \zeta}, \quad p_\zeta^2 = 0 \qquad (18)$$

is the light-cone projection of p obtained with the help of the auxiliary vector ζ. Equation (17) immediately provides us also with the explicit *inner product* for the massive spinors:

$$\bar{u}(p_1, \lambda_1)u(p_2, \lambda_2) = S(p_1, m_1, \lambda_1, p_2, m_2, \lambda_2)$$
$$\bar{u}(p_1, \lambda_1)v(p_2, \lambda_2) = S(p_1, m_1, \lambda_1, p_2, -m_2, -\lambda_2)$$
$$\bar{v}(p_1, \lambda_1)u(p_2, \lambda_2) = S(p_1, -m_1, -\lambda_1, p_2, m_2, \lambda_2)$$
$$\bar{v}(p_1, \lambda_1)v(p_2, \lambda_2) = S(p_1, -m_1, -\lambda_1, p_2, -m_2, -\lambda_2) \quad (19)$$

where

$$S(p_1, m_1, \lambda_1, p_2, m_2, \lambda_2)$$
$$= \delta_{\lambda_1, -\lambda_2}s_{\lambda_1}(p_{1\zeta}, p_{2\zeta}) + \delta_{\lambda_1, \lambda_2}$$
$$\times \left(m_1\sqrt{\frac{2\zeta p_2}{2\zeta p_1}} + m_2\sqrt{\frac{2\zeta p_1}{2\zeta p_2}} \right). \qquad (20)$$

For the sake of the subsequent discussion, let us also write the explicit basic massive spinors in the particular case when the fermion rest frame coincides with the PRF (in which $\zeta = (1, 0, 0, -1)$):

$$u(p, +) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad u(p, -) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad v(p, +) = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix},$$

$$v(p, -) = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}. \qquad (21)$$

We have omitted the trivial normalization factor $\sqrt{m}$. The convention for relative phases is inherited from the underlying massless spinors (9). In particular the important phase relations

$$S(R_2(\pi))u(p, +) = u(p, -),$$

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[8] The analogous expression in (3.9) of [6] misses factor 2. For the Levi-Civita tensor we take $\epsilon_{0123} = 1$

$$S(R_2(\pi))v(p,+) = v(p,-). \qquad (22)$$

For ultrarelativistic $p^0 \to \infty$, or for an almost massless fermion $m \to 0$, the difference between p and p_ζ is negligible and the massive spinor becomes identical with the corresponding massless spinor:

$$\lim_{m \to 0} u(p,\lambda) = \lim_{m \to 0} v(p,-\lambda) = u_\lambda(p_\zeta), \qquad (23)$$
$$\lim_{m_i \to 0} S(p_1, m_1, \lambda_1, p_2, m_2, \lambda_2) = \delta_{\lambda_1, -\lambda_2} s_{\lambda_1}(p_{1\zeta}, p_{2\zeta})$$

The validity of the above limits is restricted by the condition $p \cdot \zeta \gg m$, i.e. they fail in the special case of p parallel to ζ and such cases have to be treated one by one, with special care. In such a case, in (17), the second term, proportional to mass, survives and dominates. The same phenomenon occurs in the inner product of (20). Thus, we have to remember that the naive procedure of taking the massless fermion limit by means of straightforward omission of all terms proportional to mass can be dangerous and the limiting procedure has to be done very carefully[9].

5 Again the central question and the GPS answer

We now come to the central question of interpreting the massive fermion spinors of (14), the spin vector of (16), and finding out what the SQRF for these spinors is. The states $u(p,\pm)$ and $v(p,\pm)$ represent *pure quantum mechanical* states, which we intend to use in the Feynman rules to calculate scattering matrix elements. As we see for instance from (16), $u(p,+)$ and $u(p,-)$ have spin projections $+1/2$ and $-1/2$ on the vector $\vec{s}$ in the fermion rest frame. Thus the vector $\vec{s}$, which is *guided* by ζ, determines the z-axis of the SQRF. This is the easy part of determining the SQRF. The more difficult question is: Where are the x- and y-axes of the SQRF corresponding to $u(p,+)$ and $u(p,-)$ defined in (14)? Is this question meaningful at all? Yes, it is meaningful because a rotation $R_3(\phi)$ around the z-axis introduces a λ-dependent phase in $u'(p,\lambda)$. In fact one possible method of finding the x- and y-axes of the SQRF is to inspect $u(p,\pm)$ in the rest frame of the fermion and to adjust ϕ such that the relative phase of $u'(p,+)$ and $u'(p,-)$ is zero, i.e. such that the relation $S(R_2(\pi))u'(p,+) = u'(p,-)$ holds. This rule is, however, not very practical – we would like to have at our disposal more practical, easy to apply, rules, which determine the position of the same x- and y-axes of the SQRF for both ($\lambda = \pm$) spinors defined in (14).

The rules for determining all three spin quantization axes for $u(p,\pm)$ and $v(p,\pm)$ of (14) are the following:

– In the rest frame of the fermion, take the z-axis along $-\vec{\zeta}$.
– Place the x-axis in the plane defined by the z-axis from the previous point and the vector $\vec{\eta}$, in the same half-plane as $\vec{\eta}$.

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– With the y-axis, complete the right-handed system of coordinates.

We call the above rules the Global Positioning of Spin scheme or in short the GPS rules, and we shall call the GPS frame the SQRF determined by the GPS rules. We give here the formal proofs of the above rules.

Proof: Since all massive spinors are defined in terms of two basic massless spinors $\mathfrak{u}_+(\zeta_\downarrow)$ and $\mathfrak{u}_-(\zeta_\downarrow)$, it is sufficient to consider underlying massless spinors and to prove that after transforming them from the PRF to the GPS frame they look the same as in (8), up to a common phase factor. In other words the two basic massless spinors $\mathfrak{u}_+(\zeta_\downarrow)$ and $\mathfrak{u}_-(\zeta_\downarrow)$ get *regenerated* in the GPS frame. Without actually doing explicitly the transformation from the PRF to the GPS frame we know that because of the Dirac equation, chirality deefinition (see (7)), and because $\zeta \equiv \zeta_\downarrow = $ const $(1,0,0,-1)$ the resulting spinors have to look as follows

$$\mathfrak{u}_+(\zeta_\downarrow) = \begin{bmatrix} 0 \\ c_2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathfrak{u}_-(\zeta_\downarrow) = \begin{bmatrix} 0 \\ 0 \\ c_3 \\ 0 \end{bmatrix}, \qquad (24)$$

where c_2 and c_3 are unknown complex constants. We shall now exploit two facts: (a) in the local GPS frame, $\vec{\eta}$ lies in z-x plane (with positive x-component) and (b) the four-vector η obeys $\eta \cdot \zeta_\downarrow = 0$ and $\eta^2 = -1$. The above implies that $\eta = (-a, 1, 0, a)$, where a is a real constant, and therefore

$$\not{\eta} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2a \\ -2a & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}. \qquad (25)$$

The above completes the proof because we may exploit the relation

$$\mathfrak{u}_+(\zeta_\downarrow) = \not{\eta}\, \mathfrak{u}_-(\zeta_\downarrow) = \begin{bmatrix} 0 \\ c_3 \\ 0 \\ 0 \end{bmatrix}, \qquad (26)$$

see (9), finding out that $c_3 = c_2$, and therefore we do really get back the same looking basic spinors in the GPS frame as in the original PRF, up to a spin-independent phase factor $c = c_3 = c_2 = r \exp(i\phi)$. In fact one may advance even one step further and prove that the actual overall phase ϕ is exactly zero. We sketch a formal proof of this conjecture in the following.

Proof: Let us observe that the spinor transformations $S(B_1)$, $S(B_3)$ and $S(R_2)$, i.e. transformations induced by boosts in the x-z plane, and the rotation around the y-axis are purely real, while the $S(R_3)$ induced by the rotation around the z-axis is complex and diagonal, see (6). We shall now analyse the complete string of the Lorentz

transformations connecting the PRF and the GPS frames of a given fermion with momentum p. The first rotation $R_3(\phi_p)$ puts $\vec{p}$ in the x-z plane and introduces the following phases: $\mathfrak{u}_\lambda(\zeta_\downarrow) \to \exp(i\phi_p/2\lambda)\mathfrak{u}_\lambda(\zeta_\downarrow)$. Next, the rotation R_2 and boost B_3 and again the rotation R_2 transform us to the frame where $p = (m, 0, 0, 0)$ and $\zeta_\downarrow = (1, 0, 0, -1)$. Since the corresponding transformations of spinors are real, we know that the resulting final $\mathfrak{u}_\lambda(\zeta_\downarrow)$ preserves its complex phase! We are not yet in the GPS frame – the last transformation needed is the rotation $R_3(\phi')$ which brings $\vec{\eta}$ into the x-z plane. The resulting net phase is $\exp((i/2)(\phi_p + \phi')\lambda)$. Since we already know from the former proof of the GPS rules that there is no λ dependence in the total phase, the only possible solution is that the total change of the spinor phases is zero, $\phi_p + \phi' = 0$, modulo $4n\pi$ for some integer n, so that $\exp((i/2)(\phi_p + \phi')\lambda) = 1$ as desired. (It is also possible to give the argument, based on geometry, that $\phi_p = -\phi'$.) We have also checked numerically, using spinor transformations of (6), that the series of the above Lorentz transformation really regenerates the *basic* spinors $\mathfrak{u}_\pm(\zeta)$ in the GPS frame. For massive spinors, we have checked numerically that the massive spinors defined in (17) in the laboratory frame, when transformed properly to the GPS frame, become identical to those of (21)!

Summarizing: the important property of the GPS rules is that the basic spinors defined in (14), when transformed to the GPS rest frame of the fermion, obtain the same form (are regenerated), up to a real spin-independent, normalization constant, as the original ones, defined in (21). The Lorentz transformation from any Lorentz frame, for instance from the laboratory reference system, to the SQRF/GPS frame of every massive fermion is therefore uniquely defined. Thanks to the GPS rules, we are finally in a situation as comfortable as in the Jacob-Wick method. The difference is however clear: in the JW method, the transformation from the laboratory frame to the SQRF is the essence and the starting point of the method – it is tied up to the fermion four-momentum in the laboratory system, and the axes of that frame. In the GPS method the SQRF/GPS frame and the Lorentz transformation from the laboratory to the SQRF/GPS frame are deduced with the help of the "backward engineering" procedure guided by the auxiliary vectors ζ and η (common to all fermions), without any direct reference to the laboratory system.

6 GPS at work, τ-pair spin correlations

In the following we shall show how to apply the GPS rules in order to find out the Wick rotation [3] responsible for change of spin quantization axes between the JW and KS conventions for fermion spinors, for the spin amplitudes of the $e^-e^+ \to \tau^-\tau^+$ process.

Let us use the variant of JW helicity states defined in [3] and refer to it as JW2. The JW2 spin states are defined (quantized) in the τ^- rest frame with axes: $\widehat{z} = -\widehat{\tau^+}$ and $\widehat{y} = \widehat{\tau^+} \times \widehat{e^-}/|\widehat{\tau^+} \times \widehat{e^-}|$, and in the τ^+ rest frame with axes: $\widehat{z} = -\widehat{\tau^-}$ and $\widehat{y} = -\widehat{\tau^-} \times \widehat{e^+}/|\widehat{\tau^-} \times \widehat{e^+}|$. We denote

as $\widehat{f}$ the unit three-vector parallel to the momentum of a fermion f. The other (third) axis is always picked in such a way that the right-handed reference frame $(\widehat{x}, \widehat{y}, \widehat{z})$ is formed. With the above choice of quantization axes the double spin density matrix of the τ pair (see formula (2.6) of [9], used also in the KORALB Monte Carlo [10]) takes the form:

$$\{R_{00cd}^{JW2}\} = \mathcal{N} \tag{27}$$

$$\times \begin{bmatrix} 1+c^2+M^2s^2 & 0 & 0 & 0 \\ 0 & (1+M^2)s^2 & 0 & 2Mcs \\ 0 & 0 & (1-M^2)s^2 & 0 \\ 0 & -2Mcs & 0 & -1-c^2+M^2s^2 \end{bmatrix},$$

where $c = \cos\theta$, $s = \sin\theta$, θ is the scattering angle between e^- and τ^-, $M = 2m_\tau/s^{1/2}$ and $\mathcal{N} = 1/(1+c^2+M^2s^2)$. In order to simplify the discussion we include only s-channel photon exchange (the parity-violating Z exchange is neglected). We also neglected the electron mass. To be more precise, in [9] the spin quantization axes of $\tau^\pm$ are slightly different from JW2; let us call them the KB choice. The KB spin states are defined (quantized) in the τ^- rest frame with axes $\widehat{z} = \widehat{\tau^+}$ and $\widehat{x} = \widehat{\tau^+} \times \widehat{e^-}/|\widehat{\tau^+} \times \widehat{e^-}|$ and in the τ^+ rest frame with axes $\widehat{z} = -\widehat{\tau^-}$ and $\widehat{x} = \widehat{\tau^-} \times \widehat{e^+}/|\widehat{\tau^-} \times \widehat{e^+}|$. The transition from KB to JW2 costs additional trivial Wick rotation $R_3(\pi/2)$ for τ^+ and $R_2(\pi)R_3(\pi/2)$ for τ^-. (These rotations are merely permutations and reflection of axes.) This is what was actually done in order to get (analytically) our (27) from (2.6) of [9].

On the other hand, we calculate spin amplitudes for the same process with the basic KS spinors of (17). Using the Chisholm identity (treating electron spinors as massless) and the inner product of (20), we obtain the following result:

$$\mathcal{M}_{\lambda_1\lambda_2\mu_1\mu_2}^{\text{prod,GPS}}$$
$$= C\delta_{\lambda_1,-\lambda_2}\Big[\bar{u}(q_1,\mu_1)u(p_1,\lambda_1)\bar{v}(p_2,\lambda_2)v(q_2,\mu_2)$$
$$+\bar{u}(q_1,\mu_1)v(p_2,-\lambda_2)\bar{u}(p_1,-\lambda_1)v(q_2,\mu_2)\Big]$$
$$= C\delta_{\lambda_1,-\lambda_2}\Big[S(q_1,m_\tau,\mu_1,p_1,0,\lambda_1)$$
$$\times S(p_2,0,-\lambda_2,q_2,-m_\tau,-\mu_2)$$
$$+S(q_1,m_\tau,\mu_1,p_2,0,\lambda_2)$$
$$\times S(p_1,0,-\lambda_1,q_2,-m_\tau,-\mu_2)\Big], \tag{28}$$

where C is an overall normalization constant, unimportant for our discussion. As compared to the previous result, we lack insight into the analytical structure of the above formula; it is simply meant to be evaluated numerically. We are also unable to show analytical expressions for R_{00cd}^{GPS} – the translation from $\mathcal{M}_{\lambda_1\lambda_2\mu_1\mu_2}^{\text{prod,GPS}}$ to R_{abcd}^{GPS} using (4) has to be done numerically. In view of the nice simplicity of the JW/KB helicity amplitudes (see eg. (2.3) of [9]), one may wonder why bother about all the complications of the KS/GPS scheme. The answer is well known: the KS spinors (now with GPS supplement) will be very powerful and useful once we consider multiple photon emission
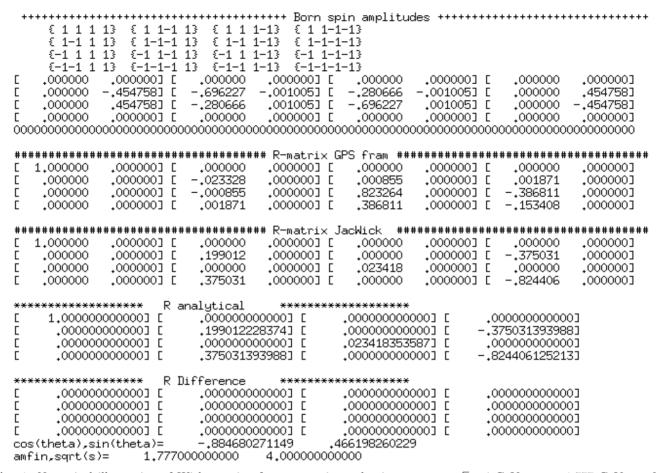
```
+++++++++++++++++++++++++++++++++++++++ Born spin amplitudes ++++++++++++++++++++++++++++++++++
      { 1  1  1  1}    { 1  1-1  1}    { 1  1  1-1}    { 1  1-1-1}
      { 1-1  1  1}    { 1-1-1  1}    { 1-1  1-1}    { 1-1-1-1}
      {-1  1  1  1}    {-1  1-1  1}    {-1  1  1-1}    {-1  1-1-1}
      {-1-1  1  1}    {-1-1-1  1}    {-1-1  1-1}    {-1-1-1-1}
[    .000000     .000000]  [    .000000     .000000]  [   -.280666    -.001005]  [    .000000     .000000]
[    .000000    -.454758]  [   -.696227    -.001005]  [   -.280666    -.001005]  [    .000000     .454758]
[    .000000     .454758]  [   -.280666     .001005]  [   -.696227     .001005]  [    .000000    -.454758]
[    .000000     .000000]  [    .000000     .000000]  [    .000000     .000000]  [    .000000     .000000]
00000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000

################################# R-matrix GPS fram ###################################
[   1.000000     .000000]  [    .000000     .000000]  [    .000000     .000000]  [    .000000     .000000]
[    .000000     .000000]  [   -.023328     .000000]  [    .000855     .000000]  [    .001871     .000000]
[    .000000     .000000]  [   -.000855     .000000]  [    .823264     .000000]  [   -.386811     .000000]
[    .000000     .000000]  [    .001871     .000000]  [    .386811     .000000]  [   -.153408     .000000]

################################# R-matrix JacWick ###################################
[   1.000000     .000000]  [    .000000     .000000]  [    .000000     .000000]  [    .000000     .000000]
[    .000000     .000000]  [    .199012     .000000]  [    .000000     .000000]  [   -.375031     .000000]
[    .000000     .000000]  [    .000000     .000000]  [    .023418     .000000]  [    .000000     .000000]
[    .000000     .000000]  [    .375031     .000000]  [    .000000     .000000]  [   -.824406     .000000]

******************     R analytical     ******************
[   1.000000000000]  [     .0000000000000]  [     .0000000000000]  [     .0000000000000]
[    .0000000000000]  [     .1990122228374]  [     .0000000000000]  [    -.3750313933988]
[    .0000000000000]  [     .0000000000000]  [     .0234183535387]  [     .0000000000000]
[    .0000000000000]  [     .3750313933988]  [     .0000000000000]  [    -.8244061253213]

******************     R Difference     ******************
[    .0000000000000]  [     .0000000000000]  [     .0000000000000]  [     .0000000000000]
[    .0000000000000]  [     .0000000000000]  [     .0000000000000]  [     .0000000000000]
[    .0000000000000]  [     .0000000000000]  [     .0000000000000]  [     .0000000000000]
[    .0000000000000]  [     .0000000000000]  [     .0000000000000]  [     .0000000000000]
cos(theta),sin(theta)=     -.884680271149      .466198260229
amfin,sqrt(s)=       1.777000000000     4.000000000000
```

Fig. 1. Numerical illustration of Wick rotation for one τ-pair production event at $\sqrt{s}$ =4 GeV, $m_\tau = 1.777$ GeV, $\cos\theta = -0.884680271149$. In the upper part we print the complex spin amplitudes $\mathcal{M}^{\mathrm{prod}}_{\lambda_1\lambda_2\mu_1\mu_2}$, with the numbering style $\{\lambda_1\lambda_2\mu_1\mu_2\}$ shown explicitly. Then, we show the R_{00cd}, $c,d = 0...4$, correlation matrix (imaginary part equal zero) as calculated in the GPS frame, using (4) and (28), and after a (double) Wick rotation to JW2 τ rest frames. On the other hand, the result marked "analytical" is obtained from (27), also in the JW2 frames. The difference of the two results in the JW2 frame is shown to be precisely zero

from fermions. In this case, the Jacob-Wick helicities are no longer so simple and useful, as they were really invented for the Born amplitudes.

The following exercise would demonstrate that our GPS rules really work: If we calculate R^{GPS}_{00cd} numerically and if we really know precisely (from the GPS rules) in which $\tau^\pm$ reference frames R^{GPS}_{00cd} is defined, then we should be able by the appropriate two non-trivial Wick rotations to get exactly the very simple R^{JW2}_{00cd} of (27). The two Wick rotations are in this case just ordinary rotations in two τ rest frames, corresponding to transformations from the GPS to the JW2 rest frames. In the following, see Fig. 1, we show the numerical results, demonstrating precisely the above operation done numerically. Indeed, we see that due to the appropriate Wick rotations the R-matrices become identical. This demonstrates clearly that in fact our GPS supplement to the KS spinor method works in the practical application. The above exercise is just a small warm-up for the paper [13], in which we apply the GPS rules to a full-scale calculation with multiple photon emissions (with exponentiation) and with decaying fermions.

7 Conclusions

In the present paper we have established a method to control the complex phase and spin quantization reference frame for the Weyl spinor technique (Kleiss-Stirling type) for massive and massless spinors. The method applies to multiphoton emission. The present work opens the way to extending the work of [5, 9, 10] (where the spin effects, photon emission and Monte Carlo event generation were successfully combined for the first time) to the case of the multiple photon emission. This is elaborated in [13]. Our paper is meant also to contribute to the discussion on the organization of the phenomenology work for the future experiments of LHC and Linear Colliders, providing relatively simple working example of the calculation involving mentioned above techniques.

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